Computer Aided Geometric Design 25 (2008) 784–791

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

# Geometry and kinematics of the Mecanum wheel

A. Gfrerrer

*Graz University of Technology, Institute of Geometry, Kopernikusgasse 24, 8010 Graz, Austria*

ARTICLE INFO

*Article history:*
Available online 4 September 2008

*Keywords:*
Mecanum wheel
Forward kinematics
Inverse kinematics

ABSTRACT

Mecanum wheels are used when omnidirectional movability of a vehicle is desired. That means that the vehicle can move along a prescribed path and at the same time rotate arbitrarily around its center. A Mecanum wheel consists of a set of rolls arranged around the wheel axis. In this paper we describe in detail the geometry of these rolls. We derive simple canonical parameterizations of the roll generating curve and the roll surface itself. These parametric representations reveal the geometry of the roll. With their help we can easily find an approximation of the roll surface by a torus for manufacture purposes. Based on the roll parametrization we study the kinematics of a vehicle featured with Mecanum wheels.



## 1. Introduction

The Mecanum wheel (Fig. 1, left) was invented by the Swedish engineer Bengt Ilon in 1973.[1] It consists of a set of $k$ congruent rolls placed symmetrically around the wheel body. The face of each roll is part of a surface of revolution $\mathcal{R}$ whose axis $b$ is skew to the wheel axis $a$. Usually an angle $\delta$ between $a$ and $b$ of $\pm 45°$ is chosen. Fig. 1, right, shows (the setup of) a mobile robot furnished with three wheels of that kind. Each of them is driven by a separate motor which gives the vehicle the three degrees of freedom necessary for an omnidirectional movement on level ground. The advantage of this architecture is that none of the wheels needs to be steerable. The wheel rolls rotate passively around their axes.

The parametrization in Dickerson and Lapin (1991) of the roll generating curve is rather involved and does not reveal the geometry of the roll. With the help of Descriptive Geometry we derive a pretty natural parametrization of this curve which also yields simple parametric representations of the roll surface and its meridian (Section 2). In Section 3 we use these parametrizations to replace the roll by an approximating torus surface. Moreover, we derive the exact velocity equation for a kinematic system with a Mecanum wheel (Section 4, Eq. (15)). With "*exact*" we mean that the position of the contact point $C$ of the roll and the terrain is also taken into account. In the literature on the kinematics of Mecanum wheels it is (as a simplification) always assumed that $C$ at any moment lies exactly beneath the wheel center (cf. for example with Viboonchaicheep et al. (2003) or Siegwart and Nourbakhsh (2004, page 59)). Using this simplification we finally study the case of a vehicle supplied with three Mecanum wheels (Section 4.1) and give a nice geometric characterization for the solvability of the forward kinematics of a such a robot.

## 2. Roll geometry of the Mecanum wheel

The roll axes of a Mecanum wheel establish a set of $k$ equidistant generators belonging to a regulus on a one-sheet hyperboloid $\mathcal{H}$ of revolution with axis $a$. If the wheel moves on a plane terrain $\pi$ its axis $a$ remaining parallel to $\pi$ then at

*E-mail address:* gfrerrer@tugraz.at.

[1] See the US patent 3,876,255 (Ilon, 1975).

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doi:10.1016/j.cagd.2008.07.008

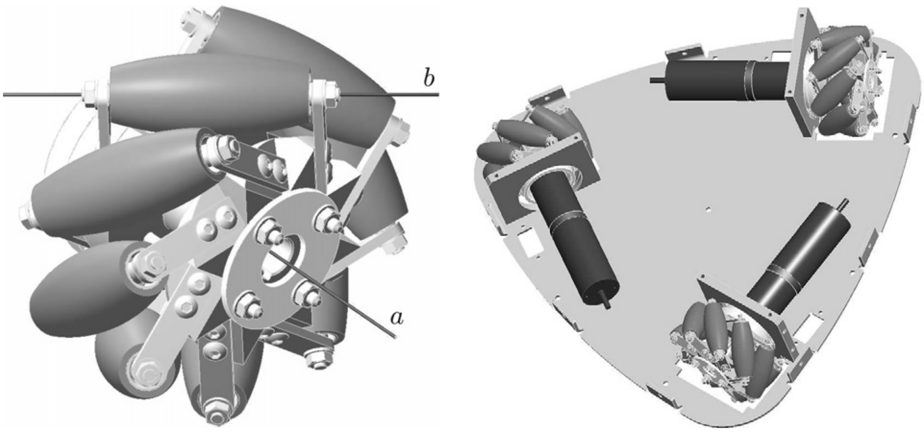


**Fig. 1.** Left: Mecanum wheel. Right: Vehicle with 3 Mecanum wheels.

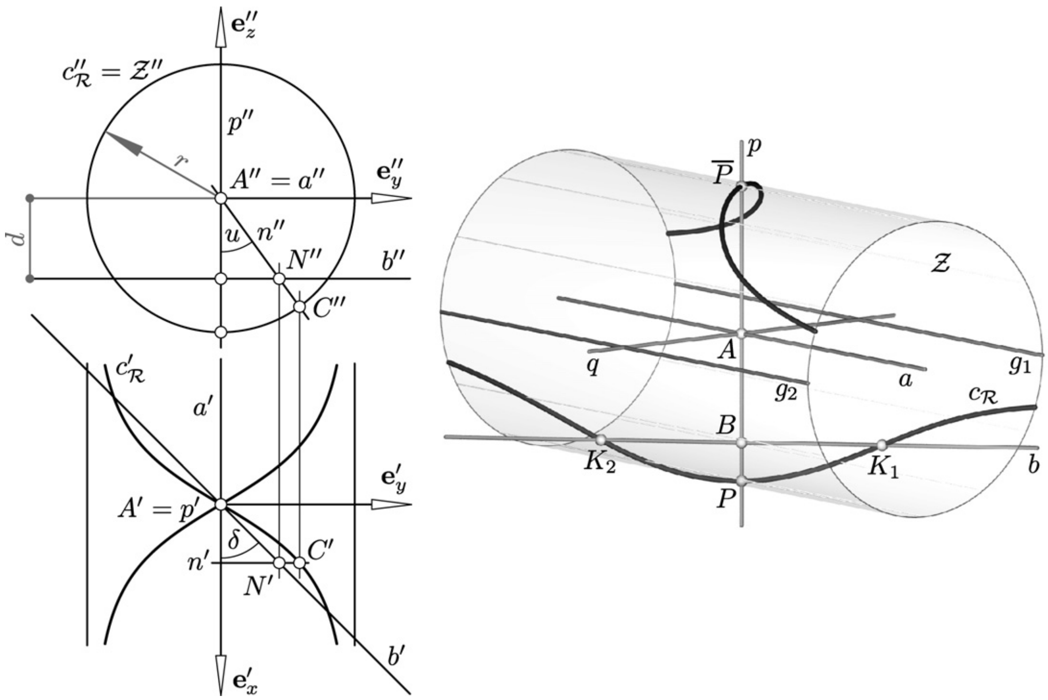


**Fig. 2.** Curve $c_\mathcal{R}$ generating the rolls.

each moment at least one roll touches the ground. Hereby a passive (non-driven) rotation around the roll axis $b$ is induced to the respective roll by the motion. Of course, it is desired to avoid vibration or jiggling of the vehicle throughout the motion, which means that the wheel axis $a$ must keep a constant distance to the plane $\pi$:

$$dist(\pi, a) = r = const. \tag{1}$$

Hence, the question is how to construct the roll surface $\mathcal{R}$ so that condition (1) is fulfilled. Fig. 2, left, shows the situation in ground view (first projection) and corresponding front view (second projection)[2] both, the wheel axis $a$ and the roll axis $b$, being parallel to the first projection plane. The rays for the second projection are parallel to $a$, i.e., the second image $a''$ of $a$ is a point.

Geometrically condition (1) means that the curve $c_\mathcal{R}$ that generates the roll surface $\mathcal{R}$ has to be a part of the cylinder $\mathcal{Z}$ of revolution with axis $a$ and radius $r$. This curve is the locus of contact points of $\mathcal{R}$ with the plane $\pi$. The roll $\mathcal{R}$ and the cylinder $\mathcal{Z}$ are tangent to each other along $c_\mathcal{R}$.

[2] We mark the first (second) image of an object by one (two) prime(s).

If $C'' \in c''_{\mathcal{R}}$ is the second image of a point $C \in c_{\mathcal{R}}$ we can easily construct its first image $C'$:

- Let $n$ denote the surface normal of $\mathcal{R}$ running through $C$. Since the circle $c''_{\mathcal{R}}$ is the second silhouette of $\mathcal{R}$ $n''$ is the diameter of $c''_{\mathcal{R}}$ containing $C''$.
- Because $n$ is a surface normal in a contour point w.r.t. the second projection it lies parallel to the plane of this projection and hence its first projection $n'$ is a horizontal line.
- As $\mathcal{R}$ is a surface of revolution $n$ has to intersect $b$ in a point $N$. In this way, the first image $n'$ of $n$ and with it the first image $C'$ of $C$ is fixed.

The above construction also yields a simple parametrization of $c_{\mathcal{R}}$. Let $p$ denote the common perpendicular of $a$ and $b$ and let $A$ be the intersection point of $p$ and $a$, i.e., the wheel center. We denote the distance and angle of $a$ and $b$ by $d$ and $\delta$ and introduce a coordinate system $S := \{A; \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ whose first and third unit vector $\mathbf{e}_x$ and $\mathbf{e}_z$ is on $a$ and $p$, respectively. As parameter we use the angle $u$ between $p$ and $n$. With the help of Fig. 2 we derive

$$\mathbf{x}(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} = \begin{pmatrix} d\cot\delta\tan u \\ r\sin u \\ -r\cos u \end{pmatrix} \tag{2}$$

as parametrization of $c_{\mathcal{R}}$. Since the two axes $a$ and $b$ are skew $\cot\delta \neq \infty$ is guaranteed.

Eq. (2) tells us that $c_{\mathcal{R}}$ in general is a rational 4th order space curve. This follows for instance by re-parameterizing $c_{\mathcal{R}}$ via $\tau = \tan\frac{u}{2}$.

Only in case of $b \perp a$ ($\delta = \pm\frac{\pi}{2}$) $c_{\mathcal{R}}$ is an ordinary circle with radius $r$. Wheels of that type are often called "*Swedish wheels*" in the literature.

Fig. 2, right, gives an impression of the curve $c_{\mathcal{R}}$ which consists of two branches. It has the axis $a$ of the cylinder $\mathcal{Z}$ and the common perpendicular $p$ of $a$ and $b$ as symmetry axes. Thus, the common perpendicular $q$ of $a$ and $p$ is also a symmetry axis of $c_{\mathcal{R}}$. The curve intersects $p$ in the points $P(0,0,-r)$, $\overline{P}(0,0,r)$ and if $\delta \neq \pm\frac{\pi}{2}$ it has the generators $g_{1,2}\ldots z=0$, $y = \pm r$ of $\mathcal{Z}$ as asymptotes. The axis $b$ and $c_{\mathcal{R}}$ meet in the common points $K_{1,2}(\pm\sqrt{r^2-d^2}\cot\delta, \pm\sqrt{r^2-d^2}, -d)$ of $b$ and $\mathcal{Z}$.

Since the curve $c_{\mathcal{R}}$ consists of two parts in case of $\delta \neq \pm\frac{\pi}{2}$, the same is true for the roll surface $\mathcal{R}$. Fig. 3 shows the part of $\mathcal{R}$ generated by the branch of $c_{\mathcal{R}}$ running through $P$. The roll $\mathcal{R}$ is a surface of revolution with axis $b$ and a symmetry plane $\sigma$ through $p$ and orthogonal to $b$. The common point $B$ of $b$ and $p$ is the symmetry center of $\mathcal{R}$. The circle $e_P \subset \sigma$ through $P$ and centered in $B$ is an equator on $\mathcal{R}$, that means it has locally maximal radius, namely $r-d$. The two points $K_{1,2}$ are conical knots on $\mathcal{R}$.

**Remark.** The surface normals $n$ used in the construction above intersect the lines $a$, $b$ and the line at infinity of the $yz$-plane. Hence, if $\delta \neq \pm\frac{\pi}{2}$ they establish a generator set on a hyperbolic paraboloid $\mathcal{P}$. As one can easily check $\mathcal{P}$ has the equation

$$xz + d\cot\delta\, y = 0.$$

The $y$-axis of the coordinate system is the axis of $\mathcal{P}$ and $A$ is its vertex. The second generator set on $\mathcal{P}$ consists of lines parallel to the $xy$-plane. Thus, $\mathcal{P}$ intersects the plane at infinity in the two lines at infinity of the $xy$- and $yz$-plane. The curve $c_{\mathcal{R}}$ is the intersection curve of $\mathcal{P}$ and the cylinder $\mathcal{Z}$. From this fact, we can see again (in a purely geometric way) that $c_{\mathcal{R}}$ is a rational fourth order curve with the asymptotes described above. The singular point of $c_{\mathcal{R}}$ is the point $X_\infty$ at infinity of the $x$-axis since this point is the vertex of the cylinder $\mathcal{R}$ and at the same time lies on the paraboloid $\mathcal{P}$. The tangent plane $\tau$ of $\mathcal{P}$ in $X_\infty$ is the $xy$-plane and intersects the cylinder in the two tangents $g_{1,2}$ of $c_{\mathcal{R}}$ in its singular point $X_\infty$.

As a surface of revolution generated by a rational curve $c_{\mathcal{R}}$ $\mathcal{R}$ itself is also rational. If $\delta \neq \pm\frac{\pi}{2}$ the algebraic order of $\mathcal{R}$ is 8, i.e., twice the order of its generating curve $c_{\mathcal{R}}$. In the special case of $\delta = \pm\frac{\pi}{2}$ the roll surface $\mathcal{R}$ is a torus whose meridian circle $c_{\mathcal{R}}$ intersects its axis $b$ in $K_{1,2}$.

Of course, for the physical roll of the Mecanum wheel only a certain part of $\mathcal{R}$ lying between the knots $K_{1,2}$ is taken.

To obtain a suitable representation of the roll surface $\mathcal{R}$ we use a new coordinate system $S^* := \{B; \mathbf{e}^*_x, \mathbf{e}^*_y, \mathbf{e}^*_z\}$ with origin in $B$, $x$-axis $x^* = b$ and the new $z$-axis coincident with the old one (Fig. 3). W.r.t. this system the curve $c_{\mathcal{R}}$ has the parametrization

$$\mathbf{x}^*(u) = \begin{pmatrix} d\frac{\cos^2\delta}{\sin\delta}\tan u + r\sin\delta\sin u \\ \cos\delta\tan u(r\cos u - d) \\ d - r\cos u \end{pmatrix}. \tag{3}$$

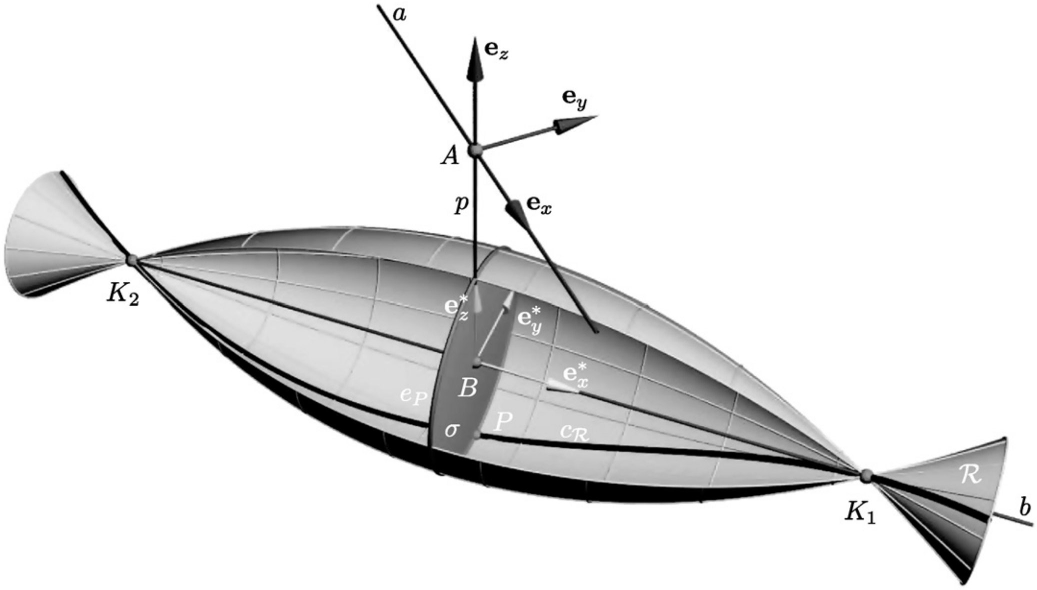


**Fig. 3.** The roll surface $\mathcal{R}$.

By rotation around the $x^*$-axis with angle $v$ we find

$$\mathbf{y}^*(u,v) = \begin{pmatrix} x^*(u,v) \\ y^*(u,v) \\ z^*(u,v) \end{pmatrix} = \begin{pmatrix} d\frac{\cos^2\delta}{\sin\delta}\tan u + r\sin\delta\sin u \\ (r\cos u - d)(\cos\delta\tan u\cos v + \sin v) \\ (r\cos u - d)(\cos\delta\tan u\sin v - \cos v) \end{pmatrix} \tag{4}$$

as parametrization of $\mathcal{R}$.

Putting $y^* = 0$ we obtain $\tan v = -\cos\delta\tan u$ which after substitution into the third line of (4) yields together with the first line a parametrization of the meridian curve $m_{\mathcal{R}}$ of the roll that lies in the $x^*z^*$-plane (parameter $u$):

$$m_{\mathcal{R}}\ldots \left\{ \begin{array}{l} x^*(u) = d\frac{\cos^2\delta}{\sin\delta}\tan u + r\sin\delta\sin u \\ z^*(u) = -\sqrt{\cos^2\delta\tan^2 u + 1}(r\cos u - d) \end{array} \right\}. \tag{5}$$

More accurately: (5) is the parametrization of **one** of the two branches of the meridian curve; the other branch is symmetric to the first one w.r.t. the axis $b = x^*$ of revolution and one gets its parametrization by changing the sign in front of the square root.

## 3. Approximation of the roll by a torus

As we have seen in the previous section the roll surface $\mathcal{R}$ of a Mecanum wheel is algebraic of order 8 generated by a fourth order space curve $c_{\mathcal{R}}$. The natural parametrizations (Eq. 4) of the roll surface and its meridian curve (Eq. 5) can be used for manufacturing the roll precisely. But since the rolls usually have a flexible rubber coat it is sufficient to use a less complicated surface which approximates $\mathcal{R}$ sufficiently accurate. For instance, one could approximate the meridian curve by a suitable conic section or a rational freeform curve. As an example, we will construct an approximating torus surface $\mathcal{T}$ for the roll $\mathcal{R}$.

**Problem 1.** Construct a torus $\mathcal{T}$ with axis $b$ so that $\mathcal{T}$ and the roll surface $\mathcal{R}$ have contact of order 2 along the equator circle $e_P$.

Due to the symmetry with respect to the plane $\sigma$ of $e_P$ the center of the wanted torus $\mathcal{T}$ must be the point $B$. Fig. 4, left, shows the situation in the $x^*z^*$-plane: One of the two meridian circles of $\mathcal{R}$ in this plane has to osculate the roll meridian $m_{\mathcal{R}}$ (Eq. (5)) at $P$. Let us denote this meridian circle by $m_{\mathcal{T}}$. Vice versa, if $m_{\mathcal{R}}$ and $m_{\mathcal{T}}$ have contact of order $k$ in $P$ than the same is true for the generated surfaces $\mathcal{R}$ and $\mathcal{T}$ along $e_P$.

The equation of $m_{\mathcal{T}}$ can be set up as

$$F(x^*, z^*) := x^{*2} + (z^* - r_l)^2 - r_m^2 = 0. \tag{6}$$

Here $r_l$ and $r_m$ denote the yet unknown radii of the center circle $l$ and of the torus meridian circle $m_{\mathcal{T}}$.

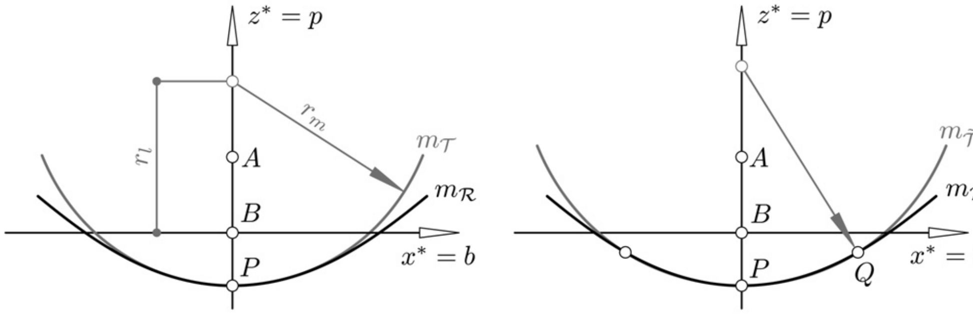


**Fig. 4.** Two ways to construct an approximating torus.

By substitution of (5) into (6) we obtain the function

$$f(u) := \sin^2 u \left( d \frac{\cos^2 \delta}{\sin \delta \cos u} + r \sin \delta \right)^2 + \left( \sqrt{\cos^2 \delta \tan^2 u + 1} (r \cos u - d) + r_l \right)^2 - r_m^2 \tag{7}$$

in $u$. Since both of the curves $m_{\mathcal{R}}$ and $m_{\mathcal{T}}$ are symmetric w.r.t. $z^*$ $f$ is an even function. Hence, all derivatives of odd order vanish at $u = 0$:

$$\left. \frac{\partial f}{\partial u} \right|_{u=0} = \left. \frac{\partial^3 f}{(\partial u)^3} \right|_{u=0} = \cdots = 0.$$

This is true for arbitrary values of $r_l$, $r_m$.

Now we determine $r_l$ and $r_m$ so that the two additional conditions

$$f(0) = (r_l - d + r)^2 - r_m^2 = 0,$$
$$\left. \frac{\partial^2 f}{(\partial u)^2} \right|_{u=0} = 2 \frac{(r \sin^2 \delta + d \cos^2 \delta)(d - r_l \sin^2 \delta)}{\sin^2 \delta} = 0$$

are fulfilled. The solution is

$$r_l = \frac{d}{\sin^2 \delta}, \tag{8}$$
$$r_m = r + d \cot^2 \delta. \tag{9}$$

For these values of $r_l$ and $r_m$ all derivatives of $f$ up to order 3 are zero at $u = 0$. Therefore, the roll meridian curve $m_{\mathcal{R}}$ and the meridian circle $m_{\mathcal{T}}$ of the torus have contact of order 3 in the point $P$. The same is true for the generated surfaces of revolution along their common equator $e_P$:

**Theorem 1.** *The roll surface $\mathcal{R}$ and the coaxial torus $\mathcal{T}$ with center circle $l \subset \sigma$ (center $B$, radius $r_l = \frac{d}{\sin^2 \delta}$) and meridian circle radius $r_m = r + d \cot^2 \delta$ have contact of order* 3 *along their common equator circle $e_P$.*

Exactly in case of Swedish wheels ($\delta = \pm \frac{\pi}{2}$) the torus surface $\mathcal{T}$ and the roll surface $\mathcal{R}$ are identical.

If especially $\delta = \pm \frac{\pi}{4}$ (the case that mainly occurs in praxis) the radii of the torus are

$$r_l = 2d, \qquad r_m = r + d.$$

Theorem 1 says that close to their common equator $e_P$ the torus $\mathcal{T}$ approximates the roll surface $\mathcal{R}$ well. Fig. 4, left, shows both roll meridian $m_{\mathcal{R}}$ and torus meridian circle $m_{\mathcal{T}}$. On the other hand this figure also reveals that at some distance from $P$ the approximation is not satisfying. So, if a Mecanum wheel is supplied with rolls of bigger length it may be advantageous to use a torus $\tilde{\mathcal{T}}$ for the approximation whose meridian circle $m_{\tilde{\mathcal{T}}}$ is tangent to $m_{\mathcal{R}}$ at $P$ and additionally contains another point $Q$ of $m_{\mathcal{R}}$ (Fig. 4, right). The point $Q$ can be computed with the help of (5). The approximation order of $\tilde{\mathcal{T}}$ along $e_P$ is only $C^1$ but at the outside regions one obtains a better approximation.

## 4. Kinematics of the Mecanum wheel

We consider a vehicle moving on level ground and furnished with Mecanum wheels like the one in (Fig. 1, right). Let us analyze the situation for one of the wheels at a certain moment $t$ (Fig. 5). Four systems are involved: the terrain $\Sigma_0$, the vehicle $\Sigma_1$, the wheel $\Sigma_2$ and the roll $\Sigma_3$ which at that moment touches the ground at a certain point $C$ (contact point). Note that this point always lies beneath the axis $a$ of the wheel $\Sigma_2$: It is the intersection point of the orthogonal projections

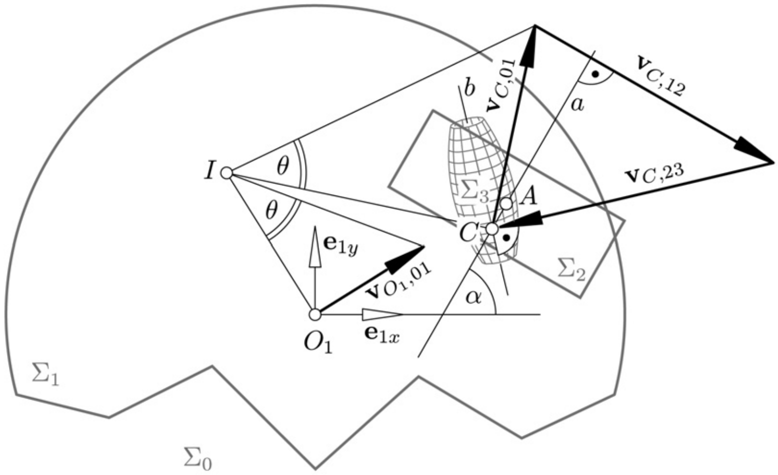


**Fig. 5.** Velocities for a vehicle with Mecanum wheels.

of the wheel axis $a$ and the roll axis $b$ in $\Sigma_0$. Only in case of $b$ being in a horizontal position $C$ lies beneath the wheel center $A$!

For the analytical description we choose an arbitrary point $O_1$ ("*vehicle center*") in $\Sigma_1$ as origin of a coordinate system $S_1 := \{O_1; \mathbf{e}_{1x}, \mathbf{e}_{1y}, \mathbf{e}_{1z}\}$ connected with the vehicle $\Sigma_1$, the $x$- and $y$-axis being parallel to the ground. The wheel center $A$ may have $x$- and $y$-coordinates $a_x$ and $a_y$ w.r.t. $S_1$ and $\alpha$ may denote the angle between $\mathbf{e}_{1x}$ and the wheel axis $a$. Then

$$\mathbf{a} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix}$$

is the direction vector of $a$. The direction vector $\mathbf{b}$ of the roll axis depends on the rotation angle $u$ of the wheel as follows:

$$\mathbf{b} = \begin{pmatrix} \cos\alpha\cos\delta - \sin\alpha\sin\delta\cos u \\ \sin\alpha\cos\delta + \cos\alpha\sin\delta\cos u \\ \sin\delta\sin u \end{pmatrix} =: \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}. \tag{10}$$

W.r.t. $S_1$ the contact point $C$ has the $x$- and $y$-coordinates

$$\left.\begin{aligned} c_x &= a_x - d\cos\alpha\cot\delta\tan u \\ c_y &= a_y - d\sin\alpha\cot\delta\tan u \end{aligned}\right\}. \tag{11}$$

In the following considerations we can neglect the $z$-coordinates since the occurring velocity vectors are all parallel to the $xy$-plane.

Let $\omega$ be the angular velocity of the motion $\Sigma_1/\Sigma_0$ (vehicle/ground) and $\mathbf{v}_{O_1,01} = (v_x, v_y)^\top$ be the velocity vector of $O_1$ for that motion at the instant $t$. Then the vectorial velocity of the contact point $C(c_x, c_y)$ w.r.t. the motion $\Sigma_1/\Sigma_0$ is[3]

$$\mathbf{v}_{C,01} = \begin{pmatrix} v_x - \omega c_y \\ v_y + \omega c_x \end{pmatrix}. \tag{12}$$

The motion $\Sigma_2/\Sigma_1$ (wheel/vehicle) is a simple rotation around the axis $a$, hence, the velocity vector of $C$ for this motion is

$$\mathbf{v}_{C,12} = \dot{u}r\begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} \tag{13}$$

where $\dot{u} = \frac{du}{dt}$ is the angular velocity of $\Sigma_2/\Sigma_1$.

The motion $\Sigma_3/\Sigma_2$ (roll/wheel) is a rotation around $b$. Thus, the instantaneous vectorial velocity $\mathbf{v}_{C,23}$ of $C$ is perpendicular to $\mathbf{b}$ (Eq. (10)):

$$\mathbf{v}_{C,23} = \lambda\begin{pmatrix} -b_y \\ b_x \end{pmatrix}. \tag{14}$$

[3] One can construct the vector $\mathbf{v}_{C,01}$ from the input $\mathbf{v}_{O_1,01}$, $\omega$ as indicated in Fig. 5. Here $I$ denotes the instantaneous pole and $\tan\theta = \omega$. Compare also with Wunderlich (1970, page 22) or Bottema and Roth (1990, page 258).

The velocity vector $\mathbf{v}_{C,03}$ of $C$ for the motion $\Sigma_3/\Sigma_0$ (roll/ground) has to be zero since the (passive) roll moves on the ground without sliding. Using the additivity rule for velocities of composed motions we obtain the condition

$$\mathbf{v}_{C,01} + \mathbf{v}_{C,12} + \mathbf{v}_{C,23} = \mathbf{v}_{C,03} = \mathbf{o} = (0,0)^\top$$

which by substitution of (12), (13), (14) yields

$$\left.\begin{aligned} r\sin\alpha\dot{u} + b_y\lambda &= v_x - \omega c_y \\ r\cos\alpha\dot{u} + b_x\lambda &= -v_y - \omega c_x \end{aligned}\right\}.$$

By elimination of $\lambda$ we get the differential equation

$$r(b_x\sin\alpha - b_y\cos\alpha)\dot{u} - b_x(v_x - \omega c_y) - b_y(v_y + \omega c_x) = 0 \tag{15}$$

ruling the connection between the vehicle motion and the wheel rotation. The terms $b_x, b_y, c_x, c_y$ in this equation are functions in $u$ according to Eqs. (10), (11) and $u$ itself, of course, depends on time $t$.

If we study the motion globally, the situation is rather complicated. While one roll of the wheel is in contact with the ground the contact point $C$ moves from the first side of the wheel to the second. When the turn is on the next roll $C$ jumps back to the first side again. It follows that $b_x(u), b_y(u), c_x(u), c_y(u)$ are functions with jump discontinuities corresponding to the changes of the rolls.[4]

This is the reason that for practical purposes[5] it is assumed that the contact point $C$ in the average lies beneath the wheel center $A$. By this simplification we can put $b_x = \cos(\alpha+\delta)$, $b_y = \sin(\alpha+\delta)$, $c_x = a_x$, $c_y = a_y$ in Eq. (15). Then we obtain

$$\dot{u} = -\frac{1}{r\sin\delta}\big[\sin(\alpha+\delta)(v_y + \omega a_x) + \cos(\alpha+\delta)(v_x - \omega a_y)\big]. \tag{16}$$

This formula allows to compute the (approximate) wheel velocity $\dot{u}$ for given vehicle velocity data $v_x, v_y, \omega$.

### 4.1. Example: Kinematics of a vehicle with three Mecanum wheels

As an example we study the case of a vehicle supplied with three Mecanum wheels with wheel centers $A_i(a_{ix}, a_{iy})$ and wheel axis angles $\alpha_i$, $i = 1,2,3$. If we denote the corresponding angular velocities of the wheels by $\omega_i$ then according to Eq. (16) we have

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = -\frac{1}{r\sin\delta}\mathbf{M}\begin{pmatrix} v_x \\ v_y \\ \omega \end{pmatrix} \tag{17}$$

with

$$\mathbf{M} = \begin{pmatrix} \cos(\alpha_1+\delta) & \sin(\alpha_1+\delta) & a_{1x}\sin(\alpha_1+\delta) - a_{1y}\cos(\alpha_1+\delta) \\ \cos(\alpha_2+\delta) & \sin(\alpha_2+\delta) & a_{2x}\sin(\alpha_2+\delta) - a_{2y}\cos(\alpha_2+\delta) \\ \cos(\alpha_3+\delta) & \sin(\alpha_3+\delta) & a_{3x}\sin(\alpha_3+\delta) - a_{3y}\cos(\alpha_3+\delta) \end{pmatrix}.$$

Eq. (17) is the solution to the inverse kinematic problem of the vehicle:

**Inverse Kinematic Problem:**
*Given*: Angular velocity $\omega$ of the vehicle $\Sigma_1$ and vectorial velocity $(v_x, v_y)^\top$ of the vehicle center $O_1$;
*Wanted*: Angular velocities $\omega_i$ of the wheels $i = 1,2,3$.

Conversely, we have the

**Forward Kinematic Problem:**
*Given*: Angular velocities $\omega_i$ of the wheels $i = 1,2,3$;
*Wanted*: Angular velocity $\omega$ of the vehicle $\Sigma_1$ and vectorial velocity $(v_x, v_y)^\top$ of the vehicle center $O_1$.

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[4] Moreover, to avoid vibration the rolls are arranged around the wheel body in a way that the silhouettes of adjacent rolls are slightly overlapping. This means that at the change of two rolls both of them are in contact with the ground for a short time interval, one close to the first side of the wheel and the other close to the second.

[5] See Viboonchaicheep et al. (2003) or Siegwart and Nourbakhsh (2004, page 59).

Clearly, a unique solution for the forward kinematic problem exists if and only if $\det \mathbf{M} \neq 0$, namely

$$\begin{pmatrix} v_x \\ v_y \\ \omega \end{pmatrix} = -r \sin\delta\, \mathbf{M}^{-1} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}. \tag{18}$$

Our simplification from above means that the axis $b_i$ of the roll that is in contact with the ground is assumed to be horizontal, roll center $B_i$ and contact point $C_i$ lying exactly beneath the wheel center $A_i(a_{ix}, a_{iy})$. In this position (the first projection of) $b_i$ has the equation

$$-\sin(\alpha_i + \delta)(x - a_{ix}) + \cos(\alpha_i + \delta)(y - a_{iy}) = 0.$$

In other words, the line coordinates of $b_i$ are

$$\big(-\sin(\alpha_i + \delta), \cos(\alpha_i + \delta), a_{ix}\sin(\alpha_i + \delta) - a_{iy}\cos(\alpha_i + \delta)\big).$$

If one multiplies the second column of $\mathbf{M}$ with $-1$ and additionally exchanges this column with the first one then the $i$-th row of the modified matrix is identical with that vector. Hence, $\det \mathbf{M} = 0$ is equivalent to the condition that the three roll axes meet in a common point $I$ which can also be at infinity. As a result we have

**Theorem 2.** *The direct kinematics of a robot with three Mecanum wheels has a unique solution if and only if the wheels are arranged so that the roll axes are not concurrent or parallel.*

In a bad wheel arrangement with the roll axes running through a common point $I$ the possible self motion of the vehicle is the rotation around $I$. In this case the three contact rolls rotate around their axes even if the wheel motors stand still. Such an unwanted self motion might be induced by slightly inclined terrain. Of course, this effect also shows up in case of vehicles with more than three Mecanum wheels.

## 5. Conclusions

In the paper I give some detailed geometric analysis of Mecanum wheels and work out natural parametrizations of the roll surface (Eq. (4)) and its meridian curve (Eq. (5)). The result can be used for manufacturing the rolls precisely. Alternatively I investigate suitable approximations of the roll surface by torus patches (Section 3).

Moreover I show that the instantaneous contact point $C$ of a roll moves from one side of the Mecanum wheel to the opposite as the wheel rotates. This is neglected in the standard literature which might be a reason for deviations between the real and the predicted motion of a vehicle on such wheels. I develop the differential equation ruling the connection between the vehicle velocity and the angular velocity of the wheel (Eq. (15)). This could be the starting point for some more accurate analysis of the kinematics of Mecanum wheel vehicles in future research work. As a drawback the formula requires the knowledge of the rotation angle function $u = u(t)$ of the wheel which may not be available in practice.

Finally, by returning to the simplified equation between the vehicle and wheel velocities, I deliver some nice geometric characterization of singular wheel constellations (Theorem 2).

## Acknowledgements

I cordially thank Mathias Brandstötter from the RoboCup Team at TU Graz for fruitful discussions and for providing the ProEngineer-files of the Mecanum wheel vehicle Krikkit$^{\mathcal{Y}}$ developed by the team. From this CAD model I generated the images in Fig. 1.

I also thank the reviewers for their help to improve the paper.

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